Continuum Modeling of Three-Dimensional Truss-like Space Structures

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A mathematical and computational analysis capability has been developed for calculating the effective mechanical properties of three-dimensional periodic truss-like structures. Two models are studied in detail. The first, called the octetruss model, is a three-dimensional extension of a two-dimensional model, and the second is a cubic model. Symmetry considerations are employed as a first step to show that the specific octetruss model has four independent constants and that the cubic model has two. The actual values of these constants are determined by averaging the contributions of each rod element to the overall structure stiffness. The individual rod member contribution to the overall stiffness is obtained by a three-dimensional coordinate transformation. The analysis shows that the effective three-dimensional elastic properties of both models are relatively close to each other.

Introduction

THE past decade has witnessed a dramatic increase in the research activities dealing with the possibility of utilizing space for commercial and scientific needs. Recently, many articles[1-4] have appeared which deal with diverse aspects of large space structures. These articles have identified various applications and also proposed novel designs of structures to meet such applications. A review of the research activities on space structures prior to 1966 has been documented in the excellent volume[5] that resulted from the International Conference on Space Structures.

It thus has become necessary to find and analyze small, lightweight three-dimensional structures that will be used easily to construct much larger space structures. It would be desirable for these structures to be isotropic in nature. However, construction requirements may make this infeasible, therefore requiring orthotropic or possibly completely anisotropic structures. The latter is also undesirable because of the added complexity to the problem. Truss-type periodic (repetitive) structures recently have been suggested and analyzed as candidates for space structures.[6,7] Here simplicity in construction coupled with large stiffness-to-density ratios will be most desirable.

The objective of the present paper is to develop analytical and computational analysis capabilities in order to generate equivalent continuum elastic properties for three-dimensional truss-like periodic structures. Broadly speaking, we outline the method as follows. Once the geometry of the repeating cell of the structure is identified, symmetry arguments are employed as a first step to identify the independent elastic coefficients. The actual values of these constants are determined by averaging the contribution of each rod element to the overall structure stiffness. The individual rod member contribution is obtained by a three-dimensional coordinate transformation. In this context, the use of the transformation relations to generate effective properties of lattice plates has been utilized by Heki.[8] In order to assess the utility of our analysis, we shall specialize our results to the following two candidate structures. The first is a three-dimensional generalization of the two-dimensional model described in Ref. 6, and the second is called a cubical model. In our subsequent analysis, we shall refer to the first as the octetruss with subscript 0 designating its properties and to the second as the cubic truss with subscript c designating its properties.

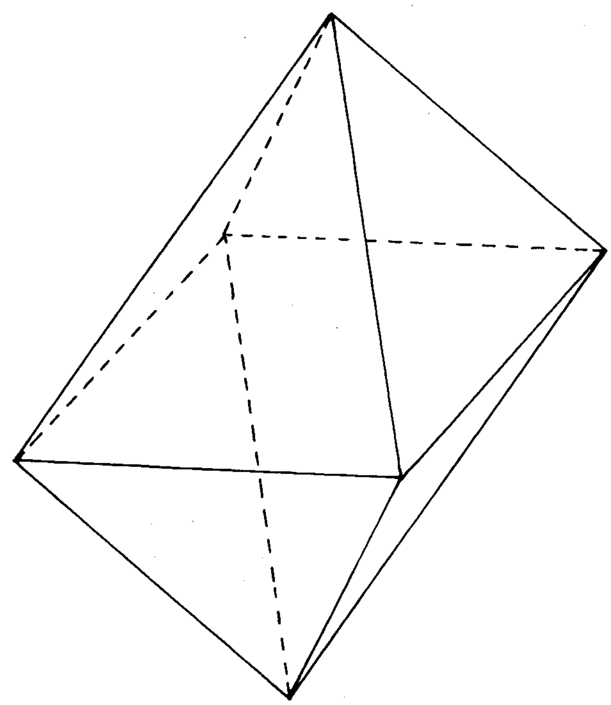

Fig. 1 Repeating cell of the octetruss models; here each side has the length L_0 and is shared by two neighboring cells.

Model Description

Octetruss Model

The smallest generating (repeating) unit cell of the octetruss structure is shown in Fig. 1. It is a diamond-like element with each of its sides having the length L_0 and being shared by two neighboring cells. The smallest substructure of the octetruss which clearly shows its complete three-dimensional symmetries is shown, respectively, in Figs. 2 and 3 with respect to two coordinate system arrangements.

A somewhat larger substructure, which conveniently displays all of the geometric characteristics of the total

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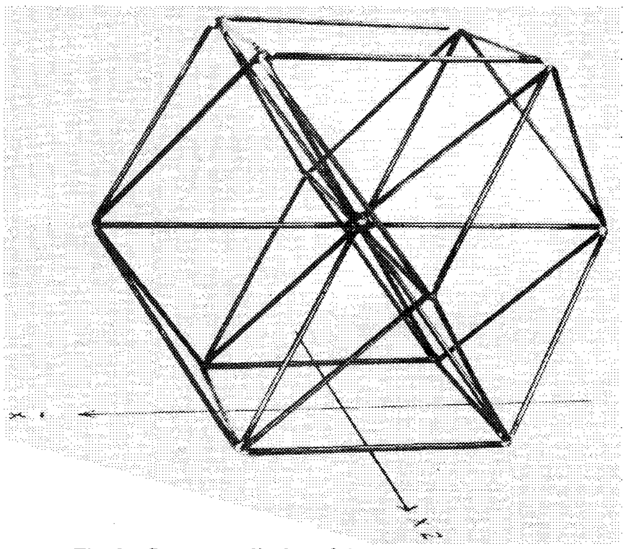

Fig. 2 Symmetry display of the octetruss structure.

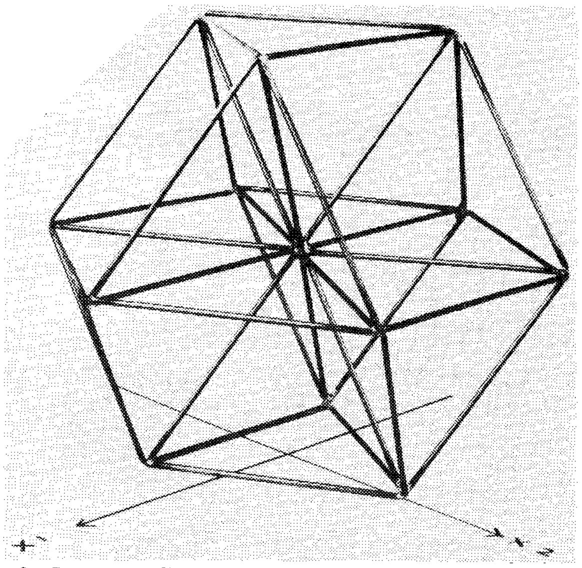

Fig. 3 Symmetry display of the octetruss structure viewed in a different coordinate system.

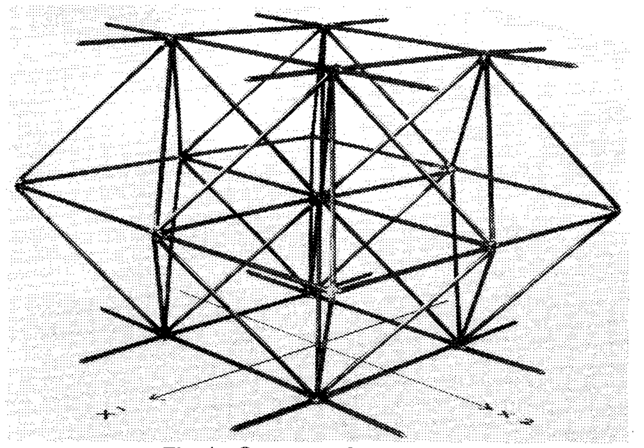

Fig. 4 Octetruss substructure.

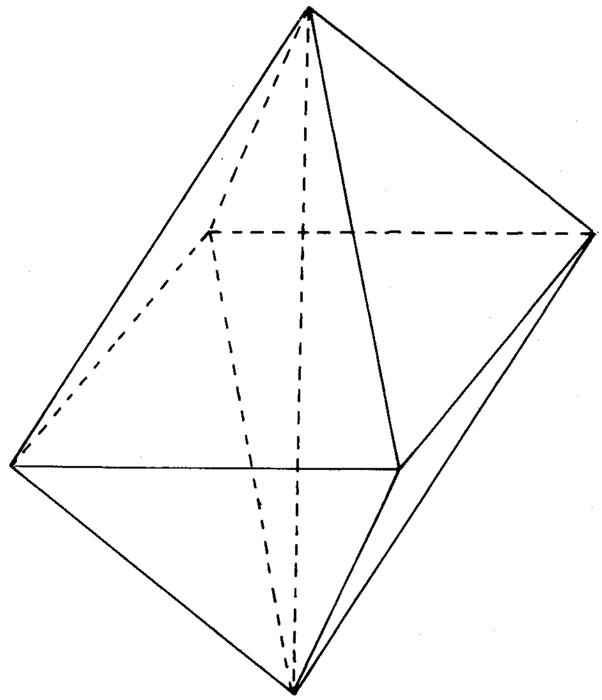

Fig. 5 Construction cell of the cubic model.

structure, is shown in Fig. 4. Inspection of this substructure shows that its volume is equivalent to the volume of eight repeating cells. Since the individual member column has the length L_0, the height of this substructure is $\sqrt{2}\,L_0$, and its base is a square whose side is $2L_0$. Thus, the effective volume of the repeating cell V_0 is obtained by dividing the volume of this substructure by eight, resulting in‡

$$V_0 = 4\sqrt{2}L_0^3/8 = L_0^3/\sqrt{2} \tag{1}$$

Since there are effectively six complete members of length L_0 in each cell, the density of the octetruss is given by

$$\rho_0 = 6\sqrt{2}A_0\gamma_0/L_0^2 \tag{2}$$

where A_0 is the cross-sectional area of the columns, and γ_0 is their material density.

Furthermore, the repeating cell will effectively have a single complete cluster joint. Thus the number of colums N_0 per unit volume of this structure is given by $N_0 = 6\sqrt{2}/L_0^3$, and the number of cluster joints n_0 per unit volume is given by $n_0 = \sqrt{2}/L_0^3$. Accordingly, there are one-sixth as many cluster joints in this model as there are individual columns.

‡This effective volume of the repeating cell is larger than its absolute volume, which is $(\sqrt{2}/3)L_0^3$.

Cubic Model

The basic construction element§ of this structure is shown in Fig. 5. It is again a diamond-like element consisting of a square base whose side length is L_c, eight inclined identical diagonals whose lengths are $\sqrt{3}\,L_c/2$, and the longer vertical diagonal with length L_c. The different members in this construction cell are shared among neighboring cells differently. The sides of the square base are shared by four cells, the inclined diagonals by three, and the vertical diagonal by one. All of these geometric constraints are also shown in Fig. 5.

A larger substructure of the cubic material which clearly displays all of its three-dimensional symmetries, its geometric characteristics, and also may serve as a repeating cell is shown in Fig. 6. It consists of a cubical truss with member columns defining the outer edges, the diagonals, and the normals from the center of the cube to the outer surfaces. It is obvious that

§Strictly speaking, in comparison with the octetruss repeating cell (Fig. 1), this element is not a repeating cell since proper rotations of this element must take place in the building process of the total structure.

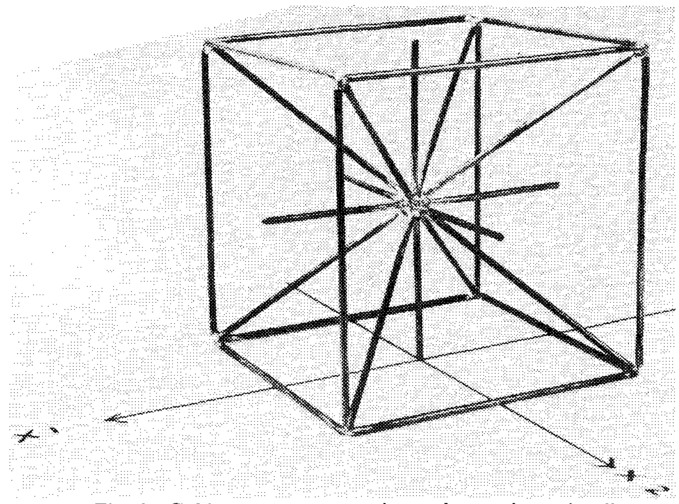

Fig. 6 Cubic structure generating and repeating unit cell.

two different column lengths are needed for its construction: those of the diagonals each having the lengths $\sqrt{3}\ L_c$, and those of the remaining members each having the lengths L_c. It is obvious that the cubical cell of the cubic structure has restrictions on its effective number of columns and cluster joints because of the sharing of some of them by neighboring cells. Specifically, each of its side columns will be shared by four repeating cells, and each of its outer cluster joints (corners) will be shared by eight cells. The remaining member columns and cluster points will belong to a single cell and thus will not be shared by other cells.

Inspection of the cubic truss of Fig. 6 reveals that its volume is equivalent to the volumes of three of the construction cells of Fig. 4. Thus the construction cell of the cubic material will have the effective volume V_c as

$$V_c = L_c^3/3 \tag{3}$$

This effective volume is the true volume of the construction cell. Now, the cubic cell has, effectively, the weight of

$$W_c = (6A_{cn} + 4\sqrt{3}A_{cd})\gamma_c L_c \tag{4}$$

where, in order to maintain generality, we allowed the diagonal members to have a different cross-sectional area A_{cd} as compared with the cross-sectional area of remaining L_c-lengthed columns A_{cn}, and γ_c is the density of the construction material. Accordingly, the density of the cubic material is

$$\rho_c = \frac{(6A_{cn} + 4\sqrt{3}A_{cd})\gamma_c}{L_c^2} \tag{5}$$

The same density ρ_c also can be obtained easily by calculating the total weight of the construction cell and dividing by its volume, Eq. (5). Finally, the number of L_c-lengthed columns and cluster joints per unit volume of the cubic material will, respectively, be $N_c = (6 + 4\sqrt{3})/L_c^3$ and $n_c = 2/L_c^3$.

Structure Stiffnesses

In the present analysis, both the octetruss and the cubical trusslike structures are considered to be composed of many repeating elements. It also is assumed that their stiffness properties may be averaged over their respective volumes. All columns making up the trusses are assumed to be pinned, so that no local bending of elements occurs.

The stress-strain equations for a general elastic body are written in the compact form

$$\sigma_i = C_{ij}\epsilon_j \tag{6}$$

where σ_i and ϵ_j are the independent six components of stress and strain, respectively. This case has 21 independent constants because the C_{ij} matrix is symmetric.

Referring to Figs. 2 and 6, which correspond to the basic cell element of the octetruss and cubic structures, respectively, we see that both structures are orthotropic (no change in their mechanical behavior when the x_1, x_2, x_3 directions are reversed). Because of this orthotropy, the following elastic coefficients vanish:

$$C_{14}, C_{15}, C_{16}, C_{24}, C_{25}, C_{26}, C_{34}, C_{35}, C_{36}, C_{45}, C_{46}, C_{56} \tag{7}$$

and the resulting stiffness matrix of Eq. (6) will now have nine independent constants. Furthermore, a 90-deg rotation in the x_1, x_2 plane of both models (Figs. 2 and 6) leaves them the same. According to Love,[9] this symmetry dictates the following further restrictions:

$$C_{22} = C_{11}, \quad C_{13} = C_{23}, \quad C_{44} = C_{55} \tag{8}$$

Thee C_{ij} matrix can finally be written in the expanded form

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{9}$$

which has six independent constants. Further symmetry consideration of the octetruss structure does not lead to any restrictions on the six independent constants. Thus, here we conclude that the octetruss structure has six independent elastic coefficients. However, for the cubic structure, further restrictions on the six coefficients is possible. Since a rotation by 90 deg in the x_1, x_3 or x_2, x_3 planes leaves the structure the same (see Fig. 6), then, similar to the previous step, the following relations must exist:

$$C_{13} = C_{12}, \quad C_{33} = C_{11}, \quad C_{66} = C_{44} \tag{10}$$

which further reduces Eq. (9) to

$$[C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \tag{11}$$

which has three distinct constants. Since further symmetry relations do not exist for the cubic structure, we finally conclude that cubic materials have three independent elastic constants.

A rather interesting alternative to the preceding symmetry considerations of the octetruss structure which led to the conclusion that such a structure has the six independent constants of Eq. (9) can be argued as follows. As we mentioned earlier, Fig. 3 shows the same basic cell element of the octetruss viewed in Fig. 2. The difference is that two different reference coordinate systems have been used. Referring to the orthogonal x_1, x_2, x_3 coordinate system of Fig. 3, the cell is aligned with the centroid of its base at the origin and one of its faces is parallel to the x_1, x_2 plane. Aligning the cell in this manner makes the x_3 axis an axis of threefold symmetry, or, in other words, the structure can be rotated about the x_3 axis by an angle of $2\pi/3$ and still give the same structure with the same material properties. As a consequence, and according to Love [Ref. 9, p. 155, Eqs. (10) and (11)], the following coefficients of the general matrix, Eq. (6), must vanish:

$$C_{16},\ C_{26},\ C_{34},\ C_{35},\ C_{36},\ C_{45} \tag{12a}$$

and the rest of the coefficients are related as follows:

$$C_{11}=C_{22},\ C_{13}=C_{23},\ C_{66}=\tfrac{1}{2}(C_{11}-C_{12}) \tag{12b}$$

$$C_{44}=C_{55},\ C_{14}=-C_{24}=C_{56},\ C_{15}=-C_{25}=-C_{46} \tag{12c}$$

and the coefficient matrix now has seven independent constants.

Further simplifications of the coefficients (12a) and (12b) are still possible. With reference to Fig. 3, a 180-deg rotation about the x_1 axis leaves the figure the same. This restriction implies the vanishing of the coefficient C_{15} [see Love,[9] Eq. (2), p. 152] and thus results in the matrix

$$[C_{ij}]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ C_{14} & -C_{14} & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{14} & C_{14} \\ 0 & 0 & 0 & 0 & C_{14} & \tfrac{1}{2}(C_{11}-C_{12}) \end{bmatrix} \tag{13}$$

which, of course, has six independent constants. Notice that the coefficients of Eq. (13) do not correspond one by one to those of Eq. (9); for example, C_{11} in Eq. (13) has no relation to C_{11} of Eq. (9). All we have shown is that, with respect to the coordinate systems of Figs. 2 and 3, the octetruss structure has six independent constants. However, the matrix in Eq. (13) has the drawback of not displaying the orthotropic character of the structure; nevertheless it displays very clearly the expected two-dimensional isotropic nature of the structure in the floor namely, the x_1, x_2 plane (see Ref. 6).

Specialization to Truss Assemblage Structures

The conclusions arrived at so far concerning the number of independent elastic constants for the octetruss and cubic structures hold equally well for any continuous or discontinuous structures so long as these structures exhibit the same corresponding symmetries. The numerical values of the appropriate C_{ij} entries thus will depend upon the specific structure under consideration. Since we are interested in analyzing truss-type structures that are constructed from one-dimensional column elements, it is expected that each member will contribute to the overall stiffness. The sum of the average contribution of each column then will lead to the final stiffness matrix. Here we indicate that our analysis assumes that the columns are perfectly straight and that they have uniform cross-sectional areas. The possible influence of the variation of the cross-sectional area, as is suggested by the nesting design concept of Bush et al.,[10] will not be addressed here.

In general, the stiffness coefficients of any anisotropic body can be transformed from one orthogonal Cartesian coordinate system x_i to another, $\bar{x}_i$ $(i=1,2,3)$ by the formulas (see, for example, Ref. 11)¶

¶There seems to be a mistake in Eq. (5.13) of Lekhnitskii[11]; the present relation, Eq. (14), is our corrected transformation.

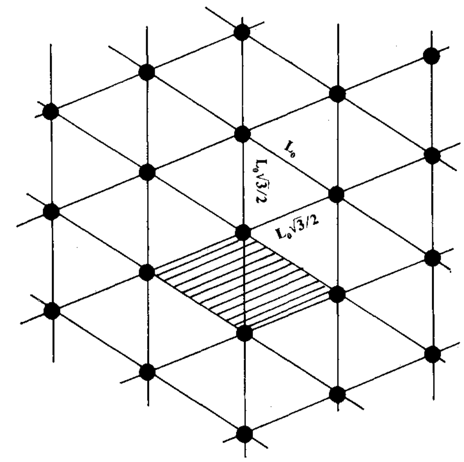

Fig. 7 The projected area of parallel members in the octetruss model.

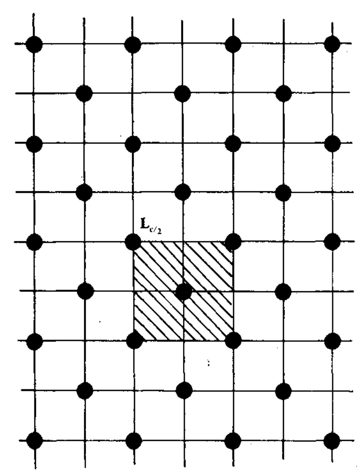

Fig. 8 The projected area of side parallel members of the cubic structure.

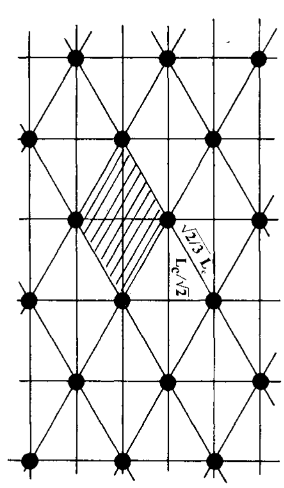

Fig. 9 The projected area of parallel diagonals of the cubic structure.

$$\bar{Q}_{ij}=\frac{1'}{\omega_i\omega_j}\sum_{m=1}^{6}\sum_{n=1}^{6}\omega_m\omega_n Q_{ij}q_{mi}q_{nj},\quad (i,\ j=1,2,3,4,5,6) \tag{14a}$$

where

$$\omega_k=\begin{cases} 1 \text{ if } k=1,2,3 \\ 2 \text{ if } k=4,5,6 \end{cases} \tag{14b}$$

and the q_{ij} are the entries of the 6×6 matrix

$$[q_{ij}] = \begin{bmatrix} \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & 2\alpha_2\alpha_3 & 2\alpha_3\alpha_1 & 2\alpha_1\alpha_2 \\ \beta_1^2 & \beta_2^2 & \beta_3^2 & 2\beta_2\beta_3 & 2\beta_3\beta_1 & 2\beta_1\beta_2 \\ \gamma_1^2 & \gamma_2^2 & \gamma_3^2 & 2\gamma_2\gamma_3 & 2\gamma_3\gamma_1 & 2\gamma_1\gamma_2 \\ \beta_1\gamma_1 & \beta_2\gamma_2 & \beta_3\gamma_3 & \beta_2\gamma_3+\beta_3\gamma_2 & \beta_1\gamma_3+\beta_3\gamma_1 & \beta_1\gamma_2+\beta_2\gamma_1 \\ \gamma_1\alpha_1 & \gamma_2\alpha_2 & \gamma_3\alpha_3 & \gamma_2\alpha_3+\gamma_3\alpha_2 & \gamma_1\alpha_3+\gamma_3\alpha_1 & \gamma_1\alpha_2+\gamma_2\alpha_1 \\ \alpha_1\beta_1 & \alpha_2\beta_2 & \alpha_3\beta_3 & \alpha_2\beta_3+\alpha_3\beta_2 & \alpha_1\beta_3+\alpha_3\beta_1 & \alpha_1\beta_2+\alpha_2\beta_1 \end{bmatrix} \tag{15a}$$

with α, β, γ defined from the direction cosines as follows:

$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline \bar{x}_1 & \alpha_1 & \beta_1 & \gamma_1 \\ \bar{x}_2 & \alpha_2 & \beta_2 & \gamma_2 \\ \bar{x}_3 & \alpha_3 & \beta_3 & \gamma_3 \end{array} \tag{15b}$$

In what follows, we shall use Eq. (14a) to specialize the coefficients of Eqs. (9) and (11) to the corresponding octetruss and cubic models under consideration. Since all columns have the single unidirectional property E, it is obvious that each set of parallel columns will define a "continuum" with a single effective unidirectional property, which we shall refer to as Q_{11}. In the context of effective modulus theories, Q_{11} will be an area-averaged modulus. Thus, the value of Q_{11} will depend not only upon the specific model under consideration but also upon the spacing of the columns in every set of parallel numbers.

Because of its complete symmetry, the octetruss model will have a single value of Q_{11} which we shall refer to as $Q_{11}^{(o)}$. On the other hand, the cubic structure has two different properties of Q_{11}: one corresponds to columns that are normal to the faces of the cube, and the other corresponds to the diagonal members. We shall refer to these two properties as $Q_{11}^{(cn)}$ and $Q_{11}^{(cd)}$, respectively, where the superscripts (cn) and (cd) designate cubic-normal and cubic-diagonal properties. Below we shall derive the quantitative values of these properties.

Effective Unidirectional Property of the Octetruss Model

The projected area of a sub-octetruss structure consisting of several repeating cells is shown in Fig. 7. On this figure, each heavy circular dot represents a column that is normal to the plane of the figure. By inspection, we see that each column occupies an effective area that is equal to the shaded area in the figure, namely, $L_0^2/\sqrt{2}$. Thus, the unidirectional property E of the individual column will be area-weighted to give

$$Q_{11}^{(o)} = \sqrt{2}\, EA_0/L_0^2 \tag{16}$$

Effective Unidirectional Properties of the Cubic Model

For the cubic structure, the unidirectional effective property $Q_{11}^{(cn)}$ can be determined easily by referring to the projected area of Fig. 8. Once again, each column effectively will occupy an area equal to half of the shaded area, namely, $L_c^2/2$. Thus, the unidirectional property E of the individual column will be area-weighted to give

$$Q_{11}^{(cn)} = 2EA_{cn}/L_c^2 \tag{17}$$

Finally, the projected area of parallel diagonals of several cubes is shown in Fig. 9. Here again each member effectively occupies an area equal to the shaded area of $L_c^2/\sqrt{3}$. It thus follows, from the foregoing that,

$$Q_{11}^{(cd)} = \sqrt{3}\, EA_{cd}/L_c^2 \tag{18}$$

The effective unidirectional properties $Q_{11}^{(o)}$, $Q_{11}^{(cn)}$, and $Q_{11}^{(cd)}$ also can be obtained easily by volume averages of E. Since the repeating cell (Fig. 1) in the octetruss has sets of two parallel half-columns, then their volume will be A_0L_0. Thus, the volume average of E can be obtained by dividing EA_0L_0 by $L_0^3/\sqrt{3}$, namely, the volume of the repeating cell as given in Eq. (1), resulting in Eq. (16). Similarly, referring to Fig. 6, we see that they are the equivalent of two complete L_c columns normal to each of the cube faces. Thus, the volume average of E with respect to such columns, namely, $Q_{11}^{(cn)}$, is obtained by dividing $2A_{cn}L_c$ by the volume of the cube, resulting in Eq. (17). Finally, since the cube in Fig. 6 has a single diagonal whose volume is $\sqrt{3}A_{cd}L_c$, then $Q_{11}^{(cd)}$ is obtained by multiplying E by the volume fraction of this diagonal, namely, $\sqrt{3}A_{cd}L_c/L_c^3$, resulting in Eq. (18).

Stiffness Matrix of the Octetruss Structure

For families of "continuums" with single unidirectional properties Q_{11}, Eq. (14a) takes the reduced form

$$\bar{Q}_{ij} = (Q_{11}/\omega_i\omega_j)\, q_{1i}q_{1j} \tag{19}$$

($i,j = 1,2,3,4,5,6$) and where, as before, the ω are defined in Eq. (14b).

In expanded matrix form, Eq. (19) becomes

$$[C_{ij}] = Q_{11} \times \begin{bmatrix} \alpha_1^4 & \alpha_1^2\alpha_2^2 & \alpha_1^2\alpha_3^2 & \alpha_1^2\alpha_2\alpha_3 & \alpha_1^3\alpha_3 & \alpha_1^3\alpha_2 \\ \alpha_1^2\alpha_2^2 & \alpha_2^4 & \alpha_2^2\alpha_3^2 & \alpha_2^3\alpha_3 & \alpha_2^2\alpha_1\alpha_3 & \alpha_2^3\alpha_1 \\ \alpha_1^2\alpha_3^2 & \alpha_2^2\alpha_3^2 & \alpha_3^4 & \alpha_2\alpha_3^3 & \alpha_3^3\alpha_1 & \alpha_3^2\alpha_1\alpha_2 \\ \alpha_1^2\alpha_2\alpha_3 & \alpha_2^3\alpha_3 & \alpha_2\alpha_3^3 & \alpha_2^2\alpha_3^2 & \alpha_3^2\alpha_1\alpha_2 & \alpha_2^2\alpha_1\alpha_3 \\ \alpha_1^3\alpha_3 & \alpha_2^2\alpha_1\alpha_3 & \alpha_1\alpha_3^3 & \alpha_3^2\alpha_1\alpha_2 & \alpha_1^2\alpha_3^2 & \alpha_1^2\alpha_2\alpha_3 \\ \alpha_1^3\alpha_2 & \alpha_2^3\alpha_1 & \alpha_3^2\alpha_1\alpha_2 & \alpha_2^2\alpha_1\alpha_3 & \alpha_1^2\alpha_2\alpha_3 & \alpha_2^2\alpha_1^2 \end{bmatrix} \tag{20}$$

Referring to Fig. 4, we recognize that from each cluster joint of the octetruss structure there emanate six double-lengthed columns. These are illustrated in Figs. 10 and 11 with respect to the coordinate systems of Figs. 2 and 3, respectively.

With reference to the coordinate system of Fig. 10, the six member columns have the following α_i, $i = 1,2,3$:

$$(1,0,0), \quad (0,1,0), \quad \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right) \tag{21a}$$

$$\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right) \tag{21b}$$

If we substitute from Eqs. (21) into Eq. (20), multiply by $Q_{11}^{(o)}$,

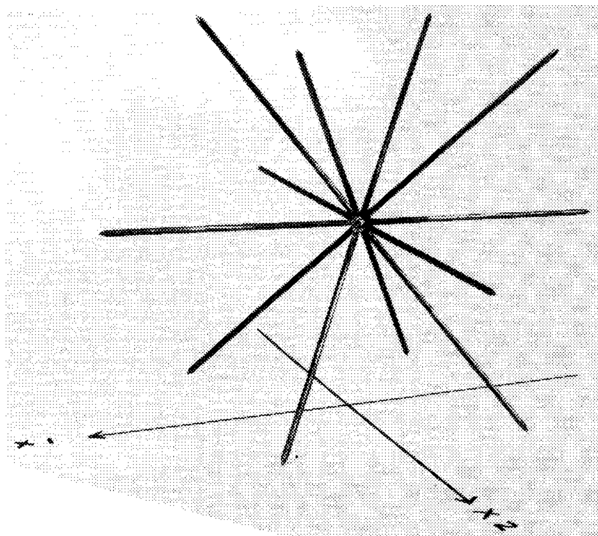

Fig. 10 Arrangement of columns emanating from every cluster point in the octetruss structure corresponding to Fig. 2.

and sum the resulting matrices, we finally find

$$[C_{ij}]=\frac{\sqrt{2}EA_0}{4L_0^2}\begin{bmatrix}5&1&2&0&0&0\\1&5&2&0&0&0\\2&2&4&0&0&0\\0&0&0&2&0&0\\0&0&0&0&2&0\\0&0&0&0&0&1\end{bmatrix}\tag{22}$$

Similarly, with respect to the coordinate system of Fig. 11, the same six member elements now will have the following α_i, $i=1,2,3$:

$$(1,0,0),\quad \left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right),\quad \left(-\frac{1}{2},\frac{\sqrt{3}}{2},0\right)\tag{23a}$$

$$\left(\frac{1}{2},\frac{\sqrt{3}}{6},\frac{\sqrt{2}}{3}\right),\quad \left(-\frac{1}{2},\frac{\sqrt{3}}{6},\frac{\sqrt{2}}{3}\right),\quad \left(0,-\frac{\sqrt{3}}{3},\frac{\sqrt{2}}{3}\right)\tag{23b}$$

which, if substituted into Eq. (20), multiplied by $Q_{II}^{(o)}$, and then summed, finally give

$$[C_{ij}]=\frac{\sqrt{2}EA_0}{4L_0^2}\begin{bmatrix}5&\frac{5}{3}&\frac{4}{3}&\frac{\sqrt{2}}{3}&0&0\\ \frac{5}{3}&5&\frac{4}{3}&-\frac{\sqrt{2}}{3}&0&0\\ \frac{4}{3}&\frac{4}{3}&\frac{16}{3}&0&0&0\\ \frac{\sqrt{2}}{3}&-\frac{\sqrt{2}}{3}&0&\frac{4}{3}&0&0\\ 0&0&0&0&\frac{4}{3}&\frac{\sqrt{2}}{3}\\ 0&0&0&0&\frac{\sqrt{2}}{3}&\frac{5}{3}\end{bmatrix}\tag{24}$$

Notice that the results, Eqs. (22) and (24), correspond, respectively, to the matrices, Eqs. (9) and (13). Notice also that the results, Eqs. (22) and (24), are not independent of

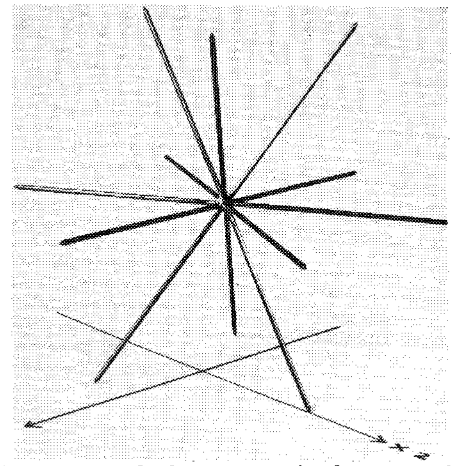

Fig. 11 Arrangement of columns emanating from every cluster point in the octetruss structure corresponding to Fig. 3.

each other; they are related by the proper transformation that relates the two coordinate systems of Figs. 10 and 11.

Stiffness Matrix of the Cubic Structure

Referring to Fig. 6, we recognize that from each cluster joint of the cubic structure there emanate seven members. A single cluster joint, together with the seven members that pass through it, is shown in Fig. 12. Notice that each member corresponds to a homogenized "continuum" with an effective unidirectional property of either $Q_{II}^{(cn)}$ or $Q_{II}^{(cd)}$. With reference to Fig. 12, the seven member elements have the following direction cosines α_i:

$$(1,0,0),\quad (0,1,0),\quad (0,0,1),\quad \left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\tag{25a}$$

$$\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right),\quad \left(\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right),\quad \left(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\tag{25b}$$

The first three of these direction cosines correspond to "continuums" whose effective unidirectional property is $Q_{II}^{(cn)}$, whereas the remaining (diagonal members) correspond to "continuums" with the effective property $Q_{II}^{(cd)}$.

Substituting from Eq. (25) into Eq. (20), multiplying the resulting matrices by their respective appropriate Q_{II}, and summing the seven matrices finally will yield the stiffness matrix:

$$[C_{ij}]=\frac{2EA_{cn}}{L_c^2}\begin{bmatrix}1+\frac{4}{9}\delta&\frac{4}{9}\delta&\frac{4}{9}\delta&0&0&0\\ \frac{4}{9}\delta&1+\frac{4}{9}\delta&\frac{4}{9}\delta&0&0&0\\ \frac{4}{9}\delta&\frac{4}{9}\delta&1+\frac{4}{9}\delta&0&0&0\\ 0&0&0&\frac{4}{9}\delta&0&0\\ 0&0&0&0&\frac{4}{9}\delta&0\\ 0&0&0&0&0&\frac{4}{9}\delta\end{bmatrix}\tag{26}$$

where we defined

$$\delta=\sqrt{3}A_{cd}/2A_{cn}\tag{27}$$

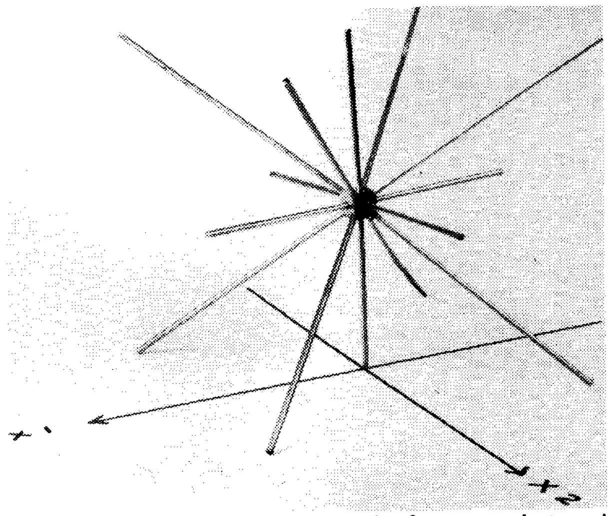

Fig. 12 Arrangement of columns emanating from every cluster point in the cubic structure corresponding to Fig. 6.

The matrix, Eq. (26) is the general stiffness matrix of the cubic structure which allows the cross-sectional area of the diagonal members to be different from those of the remaining members.

Comparison of Both Model Densities

In this subsection, we shall explore means by which a fair comparison of the densities of both the octetruss and the cubic truss structures can be made. From the onset, we shall assume that both material densities γ_0 and γ_c are equal. We also shall assume that the cross-sectional area A_0 and A_{cn} are equal but allow A_{cd} to be different. This extra degree of freedom in the design considerations, namely, allowing A_{cd} to differ from A_{cn}, will be discussed further when we calculate the effective stiffnesses of the total structures. Under the preceding conditions, the ratio of ρ_c to ρ_0 will be

$$\frac{\rho_c}{\rho_0}=\frac{(4\sqrt{3}A_{cd}/A_{cn})+6}{6\sqrt{2}}\frac{L_0^2}{L_c^2} \tag{28}$$

which also depends upon the relative lengths of both structure members. A relationship between L_0 and L_c must be established before Eq. (28) becomes meaningful. In our analysis, we propose to relate both of these lengths by imposing the condition that the basic construction cells of both models (namely, the cells corresponding to Figs. 1 and 5) have the same volume. Under this restriction, we get, by equating Eqs. (1) and (3),

$$L_0^2=(\sqrt{2}/3)^{2/3}L_c^2 \tag{29a}$$

which, if substituted into Eq. (28), yields

$$\frac{\rho_c}{\rho_0}=\left(\frac{\sqrt{2}}{3}\right)^{2/3}\frac{(4\sqrt{3}A_{cd}/A_{cn})+6}{6\sqrt{2}} \tag{29b}$$

Finally, for the special case whereby $A_{cd}=A_{cn}$ we get

$$\rho_c/\rho_0=0.92 \tag{29c}$$

If, on the other hand, we choose the relation $\sqrt{3}A_{cd}=2A_{cn}$, which, as will be shown below, happens to be an important case for calculating the stiffnesses of the cubic material, then the ratio (29c) becomes

$$\rho_c/\rho_0=0.9936\simeq 1 \tag{29d}$$

which is a somewhat surprising conclusion.

Stiffness-to-Density Ratios of Both Models

As we mentioned in the Introduction, a large stiffness-to-density ratio is most desirable in space structure. In this subsection, we shall derive such ratios for both the octetruss and the cubic truss structures. This will enable us to compare their utilities directly. To this end, dividing the stiffnesses, Eqs. (22) and (24), by the density, Eq. (2), yields the ratios

$$\left[\frac{C_{ij}}{\rho_0}\right]=\frac{E}{24\gamma_0}\begin{bmatrix}5&1&2&0&0&0\\1&5&2&0&0&0\\2&2&4&0&0&0\\0&0&0&2&0&0\\0&0&0&0&2&0\\0&0&0&0&0&1\end{bmatrix} \tag{30}$$

and

$$\left[\frac{C_{ij}}{\rho_0}\right]=\frac{E}{24\gamma_0}\begin{bmatrix}5&\frac{5}{3}&\frac{4}{3}&\frac{\sqrt{2}}{3}&0&0\\\frac{5}{3}&5&\frac{4}{3}&-\frac{\sqrt{2}}{3}&0&0\\\frac{4}{3}&\frac{4}{3}&\frac{16}{3}&0&0&0\\\frac{\sqrt{2}}{3}&-\frac{\sqrt{2}}{3}&0&\frac{4}{3}&0&0\\0&0&0&0&\frac{4}{3}&\frac{\sqrt{2}}{3}\\0&0&0&0&\frac{\sqrt{2}}{3}&\frac{5}{3}\end{bmatrix} \tag{31}$$

for the octetruss model corresponding, respectively, to the coordinate systems of Figs. 2 and 3 (also Figs. 10 and 11). Notice that both of the results, Eqs. (30) and (31), are independent of L_0 and depend only upon the quantity E/γ_0, which is the square of the wave speed in the individual single rod.

Similarly, for the cubic truss, dividing the stiffnesses, Eq. (26), by the density, Eq. (5), yields the ratio

$$\left[\frac{C_{ij}}{\rho_c}\right]=\frac{E}{(3+4\delta)\gamma_c}\times\begin{bmatrix}1+\frac{4}{9}\delta&\frac{4}{9}\delta&\frac{4}{9}\delta&0&0&0\\\frac{4}{9}\delta&1+\frac{4}{9}\delta&\frac{4}{9}\delta&0&0&0\\\frac{4}{9}\delta&\frac{4}{9}\delta&1+\frac{4}{9}\delta&0&0&0\\0&0&0&\frac{4}{9}\delta&0&0\\0&0&0&0&\frac{4}{9}\delta&0\\0&0&0&0&0&\frac{4}{9}\delta\end{bmatrix} \tag{32}$$

which is again independent of L_c and is weighted with respect to E/γ_c. If both the octetruss and the cubic truss are con-

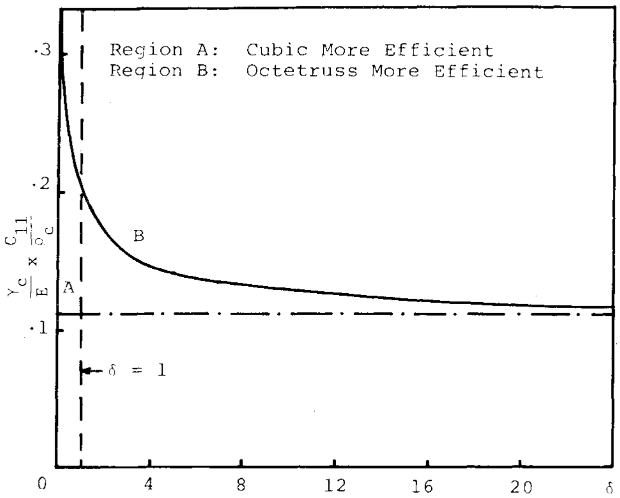

Fig. 13 Variation of (C_{11}/ρ_c) for the cubic structure with δ.

structed from the same material, then one also has $\gamma_0 = \gamma_c$. Hence the results, Eq. (30) or (31), can be compared directly with Eq. (32) once δ is specified. For the purpose of the comparison, a plot of the variation of C_{11}/ρ_c from Eq. (32) for the cubic truss is shown in Fig. 13 for various values of δ. Notice from Fig. 13 that the ratio C_{11}/ρ_c takes the maximum of $E/3\gamma_c$ when $\delta = 0$, namely, in the absence of the diagonal members (this case of mathematical interest only). As δ increases, this ratio decreases to the absolute minimum of $E/9\gamma_c$. Since from either Eq. (30) or (31) the value of C_{11}/γ_0 has the single value of $5E/24\gamma_c$ (for $\gamma_0 = \gamma_c$), then both ratios of the octetruss and cubic truss will be equal for $\delta = 27/28$. Recalling that, for $\delta = 1$, both the densities ρ_c and ρ_0 were found to be very close, here we conclude that, for δ close to unity, not only the densities of both models but also their in-plane stiffnesses are close.

Engineering Constants

The engineering constants for our models can be obtained from the corresponding C_{ij} of Eqs. (9) and (11) as follows**:

$$E_1 = E_2 = \frac{(C_{11} - C_{12})[C_{33}(C_{11} + C_{12}) - 2C_{13}^2]}{C_{11}C_{33} - C_{13}^2} \quad (33a)$$

$$E_3 = C_{33} - \frac{2C_{13}^2}{C_{11} + C_{12}} \quad (33b)$$

$$\nu_{12} = \nu_{21} = \frac{C_{12}C_{33} - C_{13}^2}{C_{11}C_{33} - C_{13}^2} \quad (33c)$$

$$\nu_{31} = \nu_{32} = \frac{C_{13}}{C_{11} + C_{12}} \quad (33d)$$

$$G_{13} = G_{23} = C_{44}, \quad G_{12} = C_{66} \quad (33e)$$

For the cubical model, the relations are given by

$$E_1 = E_2 = E_3 = C_{11} - \frac{2C_{12}^2}{C_{11} + C_{12}} \quad (34a)$$

$$\nu_{12} = \nu_{13} = \nu_{23} = \frac{C_{12}}{C_{11} + C_{12}} \quad (34b)$$

$$G_{12} = G_{13} = G_{23} = C_{44} \quad (34c)$$

**These relations are extracted from the general relations for anisotropic materials (see Ref. 11). Some relations for orthotropic materials also are given by Jones.[12]

Substituting from Eqs. (22, 24, and 26) into the preceding formulas gives the numerical values of the stiffnesses as follows:

For the Octetruss Model of Fig. 2

$$E_1 = E_2 = \frac{\sqrt{2}EA_0}{L_0^2}, \quad E_3 = \frac{2}{3}\frac{\sqrt{2}EA_0}{L_0^2} \quad (35a)$$

$$\nu_{12} = \nu_{21} = 0, \quad \nu_{31} = \nu_{32} = 0.333, \quad \nu_{13} = \nu_{23} = 0.5 \quad (35b)$$

$$G_{13} = G_{23} = \frac{1}{2}\frac{\sqrt{2}EA_0}{L_0^2}, \quad G_{12} = \frac{1}{4}\frac{\sqrt{2}EA_0}{L_0^2} \quad (35c)$$

For the Octetruss Model of Fig. 3††

$$E_1 = E_2 = \frac{\sqrt{2}EA_0}{L_0^2}, \quad E_3 = 1.2 \times \frac{\sqrt{2}EA_0}{L_0^2} \quad (36a)$$

$$\nu_{12} = \nu_{21} = 0.333, \quad \nu_{31} = \nu_{32} = 0.2, \quad \nu_{13} = \nu_{23} = 0.167 \quad (36b)$$

$$G_{23} = G_{13} = 0.3\frac{\sqrt{2}EA_0}{L_0^2}, \quad G_{12} = 0.375\frac{\sqrt{2}EA_0}{L_0^2} \quad (36c)$$

For the Cubic Model with $\delta = 1$

$$E_1 = E_2 = E_3 = 1.235 \times \frac{2EA_{cn}}{L_c^2}, \quad \nu_{12} = \nu_{23} = 0.235 \quad (37a)$$

$$G_{12} = G_{13} = G_{23} = 0.444 \times \frac{2EA_{cn}}{L_c^2} \quad (37b)$$

Discussion of Results

In specializing the general cubic materials stiffness (11) to the cubical truss structure [see Eq. (26)], two, rather than three, independent coefficients are found necessary to describe its behavior. Here, the final results, Eq. (26), suggest the further restriction that $C_{44} = C_{12}$ in Eq. (11). This restriction is not, however, incidental; it is strictly a general conclusion that can be arrived at by considering the stiffness matrix, Eq. (20), for the individual column. Referring to Eq. (20), we immediately see that‡‡ $C_{66} = C_{12}$, $C_{55} = C_{13}$, and $C_{44} = C_{23}$. Since Eq. (20) holds for each individual column, the same relations also hold for the summation matrices. Thus, we finally conclude that cubical truss-type structures have two independent elastic coefficients and similarly that the present octetruss model has four. This conclusion for the octetruss model also is confirmed numerically, as shown in Eq. (22).

The fact that cubic trusses have two independent elastic constants does not, however, imply their isotropy. For isotropy, the restriction

$$C_{66} = \tfrac{1}{2}(C_{11} - C_{12}) \quad (38)$$

must hold. Since here $C_{66} = C_{12}$, Eq. (30) requires $C_{11} = 3\,C_{12}$, which is not the case in Eq. (26).

An isotropic truss-type cubic structure can be constructed if one is willing to make restrictions on the type of materials used in the construction. For example, if the columns that are normal to the surfaces of the cube shown in Fig. 6 are allowed to have the restriction that $\delta = 9/8$, then its stiffness matrix,

††These numerical values are obtained from the corresponding values, Eqs. (35), by an appropriate transformation.

‡‡These relations are referred to in the literature as "Cauchy's" relations (see Love,[9] p. 100). They are a consequence of the assumption that any two points in the continuum interact only along their connecting line, namely, in a rod-like manner.

Eq. (26), modifies to

$$[C_{ij}] = \frac{EA_{cn}}{L_c^2} \begin{bmatrix} 3 & 1 & 1 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (39)$$

which is isotropic. Finally, by carefully inspecting the polyhedral stiffness matrix, Eq. (22), we find no material restriction cases that would result in a simplification of its mechanical behavior, especially those that would make it isotropic.

Comparison of Eqs. (35-37) reveals that little difference exists between the effective properties of both the octetruss and the cubical model, especially in the unidirectional properties E_i. Accordingly, if we are willing to accept the assumption under which the densities of both models were related in Eq. (29), then it seems, from the stiffness-to-density ratio, that the cubic model is a better candidate for the large structure.

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